A resource theory of quantum energy teleportation

Matthew Ruckman[1,*]

[1] *Unaffiliated*

(Dated: September 23, 2026)

In quantum energy teleportation (QET) a receiver, Bob, extracts energy from a strongly locally passive state after a sender, Alice, measures her part and sends him the outcome. We formulate QET as a resource theory. The free operations are Alice's instruments whose average cannot change Bob's energy (energy-non-signalling instruments), one-way classical communication, and arbitrary channels on Bob's side, with the energy that Bob's device receives booked. The QET potential $W_{\to}$, the most energy Bob can receive by free operations, is a ledger monotone: no free operation increases $W_{\to}$ plus the energy already delivered. When Alice's bond operators commute, the value of $W_{\to}$ over commuting-Kraus instruments has a closed form, which recovers Hotta's optimum for the minimal model. We prove that commuting-Kraus instruments exhaust the free class exactly when the joint eigenvalue vectors of the bond operators are in convex position. In that case $W_{\to}$ is linear in the state, the ledger holds with equality, and two-way classical communication delivers no more energy to Bob than one-way communication. Off convex position the closed form is not a monotone: in a star-coupled model, a free instrument outside the commuting class raises it 6.15-fold. The QET surplus $W_{\to} - W_{\mathrm{loc}}$ is not a monotone in either direction. For a single-qubit receiver we apply the local-ergotropy formula of Salvia, De Palma, and Giovannetti branch by branch; entropic bounds on Bob's yield are valid but loose, and we identify why. Numerical checks cover 12 500 random free operations and transverse-field Ising chains of up to 12 sites.

I. INTRODUCTION

Ground states of interacting many-body systems are, in general, strongly locally passive: no operation on a subsystem lowers the total energy [1, 2]. Hotta observed that this energy is nevertheless available to local operations aided by classical communication [3–5]. In quantum energy teleportation (QET), Alice measures a local observable and sends the outcome to Bob, who applies an outcome-dependent operation that lowers the energy near him. Alice's measurement injects an energy E_A near her, Bob extracts $E_B < E_A$ near him, and no energy travels between them faster than the classical message. QET has been demonstrated in nuclear magnetic resonance [6] and on superconducting hardware [7]; Ref. [8] reviews its applications.

Recent work varies the ingredients: the initial state [9, 10], Alice's measurement [11], and the roles of entanglement, coherence, steering, and information teleportation [12–14]. Comparing such variants requires fixing which operations are allowed and what is being spent. Resource theories supply that language: a class of free operations, the states they cannot improve, and monotones that quantify what the free operations consume [15]. This paper builds such a theory for QET.

The free operations (Sec. II) are Alice's energy-non-signalling instruments, one-way classical communication from Alice to Bob, and arbitrary channels on Bob's side. An instrument is energy-non-signalling if its average leaves Bob's energy unchanged on every state, the energetic form of no-signalling. The energy each of Bob's channels delivers to his device is booked. The monotone is the QET potential $W_{\to}$, the most energy Bob can receive by free operations. Our results are the following.

1. $W_{\to}$ is a ledger monotone: no free operation increases $W_{\to}$ plus the energy already delivered to Bob's device (Theorem 1). This is close to built in; the content lies in the next items.
2. For commuting bond operators, the potential over commuting-Kraus instruments, $W_{\to}^{\mathrm{CK}}$, is attained by Alice's projective bond measurement followed by Bob's reset to the ground state of the conditional Hamiltonian that Xie, Sajjan, and Kais call his local effective Hamiltonian [11] (Proposition 1). We state it because the next two items rest on it; for the minimal model it reproduces Hotta's optimum.
3. Commuting-Kraus instruments are the whole free class if and only if the joint eigenvalue vectors of the bond operators are in convex position (Theorem 2).
4. $W_{\to}^{\mathrm{CK}}$ is a linear functional of the state. In convex position, therefore, the ledger is an equality, and two-way classical communication delivers no more to Bob than one-way communication (Theorem 3, Corollary 2). Off convex position the closed form is not a monotone (Sec. IV B).
5. The QET surplus $W_{\to} - W_{\mathrm{loc}}$ is not a monotone under the free operations, in either direction, nor under local operations and classical communication (LOCC) (Proposition 2).
6. For a single-qubit Bob, the local-ergotropy formula of Salvia, De Palma, and Giovannetti [16] gives his optimal unitary yield on each measurement branch. We use it to show that the conditional rotation of the spin-chain protocol is optimal among unitaries,

* mruckman1@gmail.com; ORCID: 0009-0002-1723-3823

and we show why entropic bounds on the yield are loose (Sec. VII).

Everything is finite-dimensional, apart from a Gaussian analogue in Sec. VIII E. Nothing here concerns quantum fields or a thermodynamic limit. The numerical checks (Sec. VIII) use exact diagonalization of the minimal model, open transverse-field Ising chains [17] of up to 12 sites, and harmonic chains of up to four modes.

II. SETTING AND FREE OPERATIONS

A. Energy booking

Alice holds a finite-dimensional system A and Bob a system B, of dimensions d_A and d_B; the Hamiltonian H acts on $A \otimes B$. Following Hotta [5, Sec. 3], we book the interaction on Bob's side. The part of H supported on A alone,

$$H_A := \tfrac{1}{d_B}\,\mathrm{Tr}_B(H) \otimes I_B, \tag{1}$$

is Alice's energy, and $H_B^{\mathrm{loc}} := H - H_A$ is Bob's local energy operator; it contains the interaction. An operator-Schmidt decomposition gives

$$H_B^{\mathrm{loc}} = I_A \otimes B_0 + \sum_{j=1}^{m} X_j \otimes Y_j, \tag{2}$$

with the X_j traceless, Hermitian, and linearly independent, and the Y_j Hermitian and linearly independent. We call the X_j Alice's *bond operators*.

B. Free operations

(O1) Alice applies an instrument $\{K_\mu\}$ on A, with $\sum_\mu K_\mu^\dagger K_\mu = I$, whose average channel $\bar{\mathcal{A}}(\cdot) = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$ satisfies

$$(\bar{\mathcal{A}}^\dagger \otimes \mathrm{id})(H_B^{\mathrm{loc}}) = H_B^{\mathrm{loc}} \iff \bar{\mathcal{A}}^\dagger(X_j) = X_j \;\; \forall j, \tag{3}$$

where $\bar{\mathcal{A}}^\dagger(Z) = \sum_\mu K_\mu^\dagger Z K_\mu$. (The equivalence follows from applying $\bar{\mathcal{A}}^\dagger \otimes \mathrm{id}$ to Eq. (2), using $\bar{\mathcal{A}}^\dagger(I) = I$ and the linear independence of the Y_j.)

(O2) Alice sends classical messages to Bob.

(O3) Bob applies any channel $\mathcal{E}$ on B, which may depend on the messages he has received. The energy his device receives is

$$w_B(\mathcal{E},\rho) := \mathrm{Tr}[H_B^{\mathrm{loc}}\rho] - \mathrm{Tr}[H_B^{\mathrm{loc}}(\mathrm{id}\otimes\mathcal{E})(\rho)]. \tag{4}$$

It can be negative, in which case the device supplies energy.

(O4) Classical randomness may be mixed in, and classical flags discarded or relabelled.

Condition (3) says that Alice's operation, averaged over its outcomes, leaves $\langle H_B^{\mathrm{loc}}\rangle$ unchanged in every state: before Bob learns the outcome, Alice cannot change his energy. We call such instruments *energy-non-signalling*. A sufficient condition is that every Kraus operator commute with every bond operator, $[K_\mu, X_j] = 0$; we call these instruments *commuting-Kraus*. Hotta's measurement of σ_x^A in the minimal model is of this kind [5, Sec. 3].

The class is closed under composition: if $\{K_\mu\}$ and, for each μ, $\{K'_{\nu|\mu}\}$ satisfy Eq. (3), then $\sum_{\mu\nu} K_\mu^\dagger K'^\dagger_{\nu|\mu} X_j K'_{\nu|\mu} K_\mu = \sum_\mu K_\mu^\dagger X_j K_\mu = X_j$. It is a subclass of one-way LOCC. Condition (3) places no limit on the energy Alice's apparatus injects into H_A; in the minimal model that energy is Hotta's E_A. The theory tracks Bob's side. Bob's device plays the part of the battery in resource theories of work extraction: every channel on B is allowed, and the energy it exchanges with the device is booked.

C. Models

The *minimal model* [5, Sec. 3] has two qubits and, up to an additive constant,

$$H = h\sigma_z^A + h\sigma_z^B + 2k\,\sigma_x^A\sigma_x^B, \qquad h,k>0. \tag{5}$$

Here $H_B^{\mathrm{loc}} = h\sigma_z^B + 2k\sigma_x^A\sigma_x^B$, the single bond operator is $X = \sigma_x^A$, and $Y = 2k\sigma_x^B$. The *transverse-field Ising chain* is the open chain of L qubits, sites $0,\ldots,L-1$,

$$H = -J\sum_{i=0}^{L-2}\sigma_x^i\sigma_x^{i+1} - g\sum_{i=0}^{L-1}\sigma_z^i, \tag{6}$$

with $J=1$ and $g \in \{0.5, 1, 2\}$; the critical point is $g = J$. Alice's bond operators are the σ_x of her sites adjacent to Bob's. The *star model* has n_A Alice qubits and one Bob qubit,

$$H = h\sum_{i=1}^{n_A+1}\sigma_z^i + k\Big(\sum_{i\in A}\sigma_x^i\Big)\otimes\sigma_x^B, \tag{7}$$

with the single bond operator $X = \sum_{i\in A}\sigma_x^i$. It is the one model here that violates the hypothesis of Theorem 2.

III. THE QET POTENTIAL

For an instrument $\{K_\mu\}$ on A write $\tilde\rho_\mu := (K_\mu\otimes I)\rho(K_\mu\otimes I)^\dagger$ for the unnormalized branches. The *QET potential* is

$$W_\to(\rho) := \sup_{\{K_\mu\},\{\mathcal{E}_\mu\}} \sum_\mu \Big\{ \mathrm{Tr}[H_B^{\mathrm{loc}}\tilde\rho_\mu] - \mathrm{Tr}\left[H_B^{\mathrm{loc}}(\mathrm{id}\otimes\mathcal{E}_\mu)(\tilde\rho_\mu)\right]\Big\}, \tag{8}$$

the supremum running over energy-non-signalling instruments and channels $\mathcal{E}_\mu$ on B. By Eq. (3), $\sum_\mu \mathrm{Tr}[H_B^{\mathrm{loc}}\tilde\rho_\mu] = \mathrm{Tr}[H_B^{\mathrm{loc}}\rho]$, so $W_\to(\rho)$ is $\mathrm{Tr}[H_B^{\mathrm{loc}}\rho]$ minus the lowest mean energy Bob can reach by free operations. The trivial protocol shows $W_\to \geq 0$. Restricting the supremum to the trivial instrument gives Bob's local extractable energy $W_{\mathrm{loc}}(\rho)$; a strongly locally passive state has $W_{\mathrm{loc}} = 0$ [1, 2]. Restricting it to commuting-Kraus instruments gives $W_\to^{\mathrm{CK}} \leq W_\to$. We call states with $W_\to = 0$ *free*: no free protocol delivers energy to Bob from them. In every model here the ground state is not free. The vacuum is the resource.

A. A closed form for commuting bonds

Suppose the bond operators commute, with joint spectral decomposition $X_j = \sum_a x_j(a)\Pi_a$, where the Π_a are orthogonal projectors summing to I_A and the joint eigenvalue vectors $x(a) \in \mathbb{R}^m$ are distinct. Equation (2) becomes

$$H_B^{\mathrm{loc}} = \sum_a \Pi_a \otimes H^{(a)}, \qquad H^{(a)} := B_0 + \sum_j x_j(a) Y_j. \quad (9)$$

$H^{(a)}$ is the Hamiltonian Bob faces when Alice's bonds sit in block a: the local effective Hamiltonian of Ref. [11], which was introduced there for a product state.

Proposition 1 (commuting-Kraus potential)**.** *For commuting bonds,*

$$W_\to^{\mathrm{CK}}(\rho) = \sum_a \Big\{ \mathrm{Tr}[H^{(a)}\sigma_a] - p_a \lambda_{\min}(H^{(a)}) \Big\}, \quad (10)$$

where $\sigma_a := \mathrm{Tr}_A[(\Pi_a \otimes I)\rho]$ and $p_a := \mathrm{Tr}\,\sigma_a$. The value is attained by Alice's projective measurement $\{\Pi_a\}$ followed by Bob's reset of B to a ground state of $H^{(a)}$.

Proof. A commuting-Kraus K_μ commutes with every Π_a, since each Π_a is a polynomial in the X_j. By Eq. (9) and because $\mathcal{E}_\mu$ preserves trace and positivity, Bob's final energy in branch μ is $\sum_a \mathrm{Tr}[H^{(a)}\mathcal{E}_\mu(\mathrm{Tr}_A[(\Pi_a \otimes I)\tilde\rho_\mu])] \geq \sum_a \lambda_{\min}(H^{(a)})\,\mathrm{Tr}[(K_\mu^\dagger \Pi_a K_\mu \otimes I)\rho]$. Summing over μ and using $\sum_\mu K_\mu^\dagger \Pi_a K_\mu = \sum_\mu K_\mu^\dagger K_\mu \Pi_a = \Pi_a$ bounds the total final energy below by $\sum_a p_a \lambda_{\min}(H^{(a)})$, while the initial energy is $\sum_a \mathrm{Tr}[H^{(a)}\sigma_a]$. The stated protocol leaves branch a with energy $p_a\lambda_{\min}(H^{(a)})$, so it attains the bound. □

In the minimal model the blocks are the eigenspaces $a = \pm$ of σ_x^A, $H^{(\pm)} = h\sigma_z^B \pm 2k\sigma_x^B$, and $\lambda_{\min}(H^{(\pm)}) = -\sqrt{h^2+4k^2}$ for both. In the ground state [5, Sec. 3], $\langle\sigma_z^B\rangle = -h/\sqrt{h^2+k^2}$ and $\langle\sigma_x^A\sigma_x^B\rangle = -k/\sqrt{h^2+k^2}$, so $\langle g|H_B^{\mathrm{loc}}|g\rangle = -(h^2+2k^2)/\sqrt{h^2+k^2}$. Hence

$$W_\to^{\mathrm{CK}}(|g\rangle) = \sqrt{h^2+4k^2} - \frac{h^2+2k^2}{\sqrt{h^2+k^2}}. \quad (11)$$

Since $(h^2+4k^2)(h^2+k^2) = (h^2+2k^2)^2 + h^2k^2$, this equals Hotta's $E_B^{\max} = \frac{h^2+2k^2}{\sqrt{h^2+k^2}}\big[\sqrt{1+h^2k^2/(h^2+2k^2)^2} - 1\big]$ [5, Eq. (11)]. Alice holds a qubit, so commuting-Kraus instruments are her whole free class (Theorem 2), and $W_\to(|g\rangle) = E_B^{\max}$. Hotta's protocol is therefore optimal among all free protocols, and $E_B^{\max}$ is the resource it spends. Hotta's injected energy is $E_A = h^2/\sqrt{h^2+k^2}$ [5, Eq. (8)], so with $r := k/h$ and $x := 1+2r^2$,

$$\frac{E_B^{\max}}{E_A} = \sqrt{x^2+r^2} - x < \frac{r^2}{2x} < \frac{1}{4}, \quad (12)$$

using $\sqrt{x^2+r^2} < x + r^2/(2x)$. In the minimal model Bob never receives a quarter of what Alice injects; the ratio approaches $1/4$ only as $k/h \to \infty$ (Fig. 1). Equation (12) is an elementary consequence of Hotta's two formulas.

B. Monotonicity

Theorem 1 (ledger monotonicity)**.** *For every state ρ:*

(a) for every energy-non-signalling instrument on A, with branches ρ_μ of probability p_μ,

$$\sum_\mu p_\mu W_\to(\rho_\mu) \leq W_\to(\rho); \quad (13)$$

(b) for every channel $\mathcal{E}$ on B,

$$W_\to\big((\mathrm{id}\otimes\mathcal{E})(\rho)\big) + w_B(\mathcal{E},\rho) \leq W_\to(\rho). \quad (14)$$

Both statements hold for $W_\to^{\mathrm{CK}}$ when (a) is restricted to commuting-Kraus instruments.

The proof is in Appendix A. Inequality (13) is strong monotonicity, i.e., monotonicity on average over branches [15]. Because $W_\to$ is a supremum of linear functionals, it is convex, so it is also non-increasing under the average channel and under mixing.

Corollary 1 (device ledger)**.** *Let D be the energy delivered to Bob's device so far. Along any free protocol, the expected value of $W_\to(\text{state}) + D$ does not increase. Hence the energy any free protocol delivers to Bob's device is at most $W_\to(\rho)$; $W_\to$ does not increase under free operations that draw no energy from the device; and a free operation can raise $W_\to$ only by drawing energy from the device, and by at most the amount drawn.*

In the minimal model the ledger reads as follows. Alice's measurement leaves $W_\to = E_B^{\max}$ on average (Theorem 3 below makes this an equality). After the measurement the energy is locally extractable on each branch, because Alice's state lies in a single block. Bob's rotation delivers $E_B^{\max}$ and leaves branches with $W_\to = 0$. Corollary 1 is not a statement about $W_\to$ alone. Bob may excite his own system; that raises $W_\to$, but only by the energy he put in.

IV. THE FREE CLASS

Theorem 2 (free class)**.** *Let the bond operators commute, with blocks Π_a and joint eigenvalue vectors $x(a)$. Every energy-non-signalling instrument on A is commuting-Kraus if and only if every $x(a)$ is an extreme point of $\mathrm{conv}\{x(a')\}_{a'}$, i.e., the $x(a)$ are in convex position.*

The key identity is the following. Set $M_{ab} := \sum_\mu \Pi_b K_\mu^\dagger \Pi_a K_\mu \Pi_b \geq 0$. Sandwiching completeness and Eq. (3) between Π_b's gives

$$\sum_a M_{ab} = \Pi_b, \qquad \sum_a x(a) M_{ab} = x(b)\,\Pi_b. \tag{15}$$

For a unit vector v in the range of Π_b, the numbers $p_a = \langle v|M_{ab}|v\rangle$ thus form a probability distribution with barycentre $x(b)$. If $x(b)$ is extreme, $p_a = 0$ for $a \neq b$. Hence $M_{ab} = 0$, i.e., $\Pi_a K_\mu \Pi_b = 0$, and every K_μ is block diagonal. Conversely, suppose $x(b) = \sum_{a\neq b} \lambda_a x(a)$ for a probability vector λ. Choose unit vectors $|b\rangle$ in the range of Π_b and $|a\rangle$ in the range of each Π_a. Then

$$K_0 = I - |b\rangle\langle b|, \qquad K_a = \sqrt{\lambda_a}\,|a\rangle\langle b| \tag{16}$$

is complete and satisfies Eq. (3), since $\sum K^\dagger X_j K = X_j - x_j(b)|b\rangle\langle b| + \sum_a \lambda_a x_j(a)|b\rangle\langle b| = X_j$. Yet $\Pi_a K_a \Pi_b \neq 0$. The witness (16) is non-unital: $\bar{\mathcal{A}}(I) = I - |b\rangle\langle b| + \sum_a \lambda_a |a\rangle\langle a| \neq I$. That is unavoidable. For a unital instrument, $\bar{\mathcal{A}}^\dagger$ is a unital, trace-preserving channel, and its fixed points form the commutant of the Kraus operators [18]. So a unital energy-non-signalling instrument is commuting-Kraus whatever the spectrum, and convex position is exactly what the non-unital instruments need. The complete proof is in Appendix A.

Which models are in convex position? A single-qubit A always is. Commuting traceless Hermitian 2×2 matrices are proportional to one $\boldsymbol{n}\cdot\boldsymbol{\sigma}$, so the points are $\pm v$. Every region of the Ising chain (6) is too, since its bond operators are σ_x's whose joint eigenvalues are the vertices of a box. The star model (7) with $n_A \geq 2$ is not: $X = \sum_i \sigma_x^i$ has the eigenvalues $n_A, n_A - 2, \ldots, -n_A$, and all but the two outermost are interior.

A. A certificate

Everything in Eq. (15) depends on the instrument only through its average channel, $M_{ab} = \Pi_b \bar{\mathcal{A}}^\dagger(\Pi_a)\Pi_b$, and Eq. (3) is linear in the Choi operator J of $\bar{\mathcal{A}}$. Maximizing the weight outside the block-diagonal,

$$w_{\text{off}}(J) := \sum_{a\neq b} \mathrm{Tr}\, M_{ab}, \tag{17}$$

subject to $J \geq 0$, trace preservation, and Eq. (3), is therefore a semidefinite program. Its optimum is zero exactly when every energy-non-signalling instrument is commuting-Kraus. The optimum is in fact $\sum_{b\ \text{non-extreme}} \mathrm{rank}\,\Pi_b$. It is at most that, because by Eq. (15) the extreme blocks contribute nothing and block b contributes at most $\mathrm{Tr}\,\Pi_b$. Applying the witness (16) to every vector of an orthonormal basis of every non-extreme block attains it; take $\lambda_b = 0$, which is possible because a non-extreme $x(b)$ lies in the hull of the other points. Table I lists the solver's optima. They agree with the analytic verdict on every row and equal the total rank of the non-extreme blocks: 2 and 6 on the two star models.

TABLE I. The free class of each model. Ranks are those of the joint eigenprojectors Π_a of Alice's bond operators. $w_{\text{off}}^{\text{SDP}}$ is the solver optimum of Eq. (17). The last column is the total rank of the non-extreme blocks, which equals the exact optimum. Minimal-model rows use $(h,k) = (1, \frac{1}{2})$ with Alice at qubit 0 and $(0.3, 2)$ with Alice at qubit 1; Alice's sites are listed. Ising rows use $g = 1$. Star rows use $(h,k) = (1, \frac{1}{2})$.

Model	Alice	Ranks	Convex	$w_{\text{off}}^{\text{SDP}}$	Exact
Minimal	$\{0\}$	1,1	yes	2×10^{-16}	0
Minimal	$\{1\}$	1,1	yes	1×10^{-15}	0
Ising $L=5$	$\{0\}$	1,1	yes	1×10^{-15}	0
Ising $L=5$	$\{1,2\}$	1,1,1,1	yes	-8×10^{-11}	0
Ising $L=6$	$\{2,3\}$	1,1,1,1	yes	-8×10^{-11}	0
Star $n_A=2$	$\{0,1\}$	1,2,1	no	2.000	2
Star $n_A=3$	$\{0,1,2\}$	1,3,3,1	no	6.000	6

B. Off convex position

Take the star model with $n_A = 2$, $(h,k) = (1, \frac{1}{2})$, and its ground state. There $W_\to^{\text{CK}}(|g\rangle) = 0.036634$. Apply the witness (16), whose energy-signalling defect is 9×10^{-16}, then the optimal commuting-Kraus protocol on each branch. Bob receives 0.225331, which is 6.15 times as much. With only the witness outcome, Bob's local extraction is already 0.188697 (5.15 times $W_\to^{\text{CK}}$). Hence $W_\to(|g\rangle) \geq 0.225331 > W_\to^{\text{CK}}(|g\rangle)$ there. The closed form (10) is not the potential, and it is not a monotone under the free class: its average over the witness's branches exceeds its initial value by 0.188697. For $n_A = 3$ a witness exists, but this particular one extracts nothing measurable (6×10^{-14}). The optimal free instrument off convex position is not known.

V. LINEARITY AND TWO-WAY COMMUNICATION

Theorem 3 (linearity and the exact ledger)**.** *For commuting bonds, $W_\to^{\text{CK}}(\rho) = \mathrm{Tr}[W_{\text{op}}\rho]$ with*

$$W_{\text{op}} := \sum_a \Pi_a \otimes \big(H^{(a)} - \lambda_{\min}(H^{(a)})I\big) = H_B^{\text{loc}} + G, \tag{18}$$

where $G := -\sum_a \lambda_{\min}(H^{(a)})\,\Pi_a \otimes I_B$ is supported on A. Consequently:

(a) for every instrument $\{\mathcal{L}_\nu\}$ on B, with branches ρ_ν of probability p_ν, $\sum_\nu p_\nu W_\to^{\mathrm{CK}}(\rho_\nu) = W_\to^{\mathrm{CK}}(\rho) - w_B$, where $w_B = \mathrm{Tr}[H_B^{\mathrm{loc}}\rho] - \sum_\nu \mathrm{Tr}[H_B^{\mathrm{loc}}(\mathrm{id}\otimes\mathcal{L}_\nu)(\rho)]$ is the energy the instrument delivers to Bob's device;

(b) for every commuting-Kraus instrument on A, $\sum_\mu p_\mu W_\to^{\mathrm{CK}}(\rho_\mu) = W_\to^{\mathrm{CK}}(\rho)$.

If the $x(a)$ are in convex position, then $W_\to = W_\to^{\mathrm{CK}}$ and (a) and (b) hold for all free operations: the ledger of Corollary 1 is conserved exactly.

Proof. Linearity follows from Eqs. (10) and (9). For (a), the sum over branches of $\mathrm{Tr}[G(\mathrm{id}\otimes\mathcal{L}_\nu)(\rho)]$ equals $\mathrm{Tr}[G\rho]$ because G acts on A alone and $\sum_\nu \mathcal{L}_\nu$ is trace preserving; the H_B^{loc} part changes by $-w_B$. For (b), commuting-Kraus instruments satisfy $\sum_\mu K_\mu^\dagger \Pi_a K_\mu = \Pi_a$, which preserves G, and Eq. (3), which preserves H_B^{loc}. The last sentence follows from Theorem 2. □

Corollary 2 (two-way communication gives Bob nothing)**.** *Consider protocols of finitely many rounds. In each round either Alice applies a commuting-Kraus instrument on A or Bob applies an instrument on B; the outcome is broadcast, and every choice may depend on all earlier outcomes. The expected energy delivered to Bob's device is $W_\to^{\mathrm{CK}}(\rho) - \mathbb{E}[W_\to^{\mathrm{CK}}(\rho_{\mathrm{fin}})] \le W_\to^{\mathrm{CK}}(\rho)$, and one-way communication attains it. In convex position this holds for all free instruments on A, and the two-way value equals $W_\to(\rho)$.*

Proof. By Theorem 3, the expectation of $W_\to^{\mathrm{CK}}(\text{state}) + D$ is the same after every round. It starts at $W_\to^{\mathrm{CK}}(\rho)$ and ends at no less than D, since $W_\to^{\mathrm{CK}} \ge 0$. Proposition 1 attains it in one round. □

The mechanism is linearity. Because $W_\to^{\mathrm{CK}}$ is linear, what Bob learns by measuring cannot raise its expectation, and Alice's commuting-Kraus instruments conserve it whatever they are conditioned on. Bob's own instruments pay for every rise with an equal injection.

The scope matters. Off convex position, part (b) fails for free instruments outside the commuting class (Sec. IV B), and the failure already occurs with one-way communication. Corollary 2 then says nothing about the true potential $W_\to$. Whether two-way communication can beat one-way communication when the bond spectrum is not in convex position is open. We also restrict to finitely many rounds; the round number of LOCC protocols is itself a resource [15].

Remark 1 (crediting both sides). Two-way protocols can deliver more energy in total if a device on Alice's side is also credited. To see this, book the interaction half to each side, which leaves Eq. (3) and hence Alice's free class unchanged. In each round one party measures its finest free instrument and the other cashes out optimally, with the measuring party chosen to maximize the total. The total delivered then grows with the number of rounds r (Table II), reaching $20.5\,W_\to$ at $r = 3$ for $(h,k) = (1,\frac{1}{2})$. But the measurements inject exactly as much more. The net energy taken from the system, delivered minus injected, is -0.8219 for $r = 1,2,3$. It is round-independent to 2×10^{-13} in all seven cases, and negative, as it must be for a ground state, which is passive [19]. The extra delivery is the measuring apparatus's energy passing through the system.

TABLE II. Two-way protocols with both sides' devices credited. $W_\to$ is the one-way potential; the next three columns give the total delivered after $r = 1,2,3$ rounds; the net is delivered minus injected, which is the same for every r. Minimal-model rows are labelled by (h,k); Ising rows have $g = 1$ and Alice at site 0. The $L = 7,8$ rows were run to two rounds.

Case	$W_\to$	$r=1$	$r=2$	$r=3$	Net
$(1,\frac{1}{2})$	0.0726	0.0726	0.7797	1.4868	−0.8219
$(0.3,2)$	0.0110	0.0110	0.0334	0.0559	−0.0335
$(2,2)$	0.2295	0.2295	1.1239	2.0183	−1.1847
$L=5$	0.1447	0.1447	0.7410	1.3832	−0.7129
$L=6$	0.1546	0.1546	0.7476	1.3861	−0.7005
$L=7$	0.1620	0.1620	0.7530	—	−0.6915
$L=8$	0.1677	0.1677	0.7575	—	−0.6848

VI. THE SURPLUS AND THE NEGATIVITY

A natural candidate for "what classical communication buys" is the surplus $Q := W_\to - W_{\mathrm{loc}} \ge 0$. It is not a monotone.

Proposition 2 (the surplus)**.** *Let the bonds commute, let Bob hold all of B, and restrict Alice to commuting-Kraus instruments. Then:*

(a) Q does not increase on average under Alice's instruments, and does not decrease under channels on B.

(b) Q can increase under a local unitary on A, an LOCC operation outside the free class. In the minimal model, $\rho_1 = |+\rangle\langle+| \otimes |0\rangle\langle 0|$ has $Q = 0$. The unitary $U_A : |+\rangle \mapsto |0\rangle$ maps it to $\rho_2 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, which has $Q = \sqrt{h^2+4k^2} - h$.

(c) Q can increase under a free channel on B that delivers energy to Bob's device. The state $\rho_3 = \frac{1}{2}(|+\rangle\langle+| \otimes |0\rangle\langle 0| + |-\rangle\langle-| \otimes |1\rangle\langle 1|)$ has $Q = 0$. Bob's reset of B to $|1\rangle$ delivers h and yields $\rho_5 = \frac{I}{2} \otimes |1\rangle\langle 1|$, which has $Q = \sqrt{h^2+4k^2} - h$.

Here $|\pm\rangle$ are the eigenvectors of σ_x, and $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$.

The proof is in Appendix A. Q is therefore neither non-increasing nor non-decreasing under the free operations, and it is not an LOCC monotone. In (c) Bob extracts energy and Q rises, so Q is not what Bob spends; $W_{\to}$ is. In (c) $W_{\to}$ falls from $\sqrt{h^2+4k^2}$ to $\sqrt{h^2+4k^2}-h$, by exactly the delivered h, as Theorem 3(a) requires. At $(h,k)=(1,\frac{1}{2})$ the rises in (b) and (c) are both $\sqrt{2}-1=0.41421$, and the code reproduces them. Over 60 random commuting-Kraus instruments, Q never rose by more than 6×10^{-14}. Over 60 random channels on B it never fell by more than 9×10^{-14}.

The negativity $N(\rho)=(\|\rho^{T_A}\|_1-1)/2$ is an LOCC monotone [20], and hence a monotone under the free operations, which are a subclass of LOCC. It is not what Bob spends, for three reasons. In the minimal model Alice's measurement leaves product branches, with $N=0$, that still carry $W_{\to}=E_B^{\max}$. The state ρ_2 above is a product state with $Q>0$. On the chains, the two-site negativity $N(\rho_{AB})$ vanishes for $d\geq 3$ while Bob's yield stays positive (Fig. 2); the correlations Bob uses at those distances are carried by the rest of the chain. Hotta derived relations between teleported energy and ground-state entanglement for the minimal model [21].

VII. A SINGLE-QUBIT RECEIVER

On the chains Bob holds one site b at distance d from Alice. An operation on b changes only the terms of H that touch b. We evaluate energies with an operator H_R on $R=\{b-1,b,b+1\}$ that contains every such term: the part of the three sites' local energy densities that is supported on R. Let τ_μ be the conditional state of R after Alice's outcome μ. Bob's yield from unitaries U_μ on b is $E_B=\sum_\mu p_\mu\big(\operatorname{Tr}[H_R\tau_\mu]-\operatorname{Tr}[H_RU_\mu\tau_\mu U_\mu^\dagger]\big)$.

A. The exact unitary yield

Write $H_R=I_b\otimes C_0+\sum_{i=1}^3\sigma_i\otimes C_i$ and let $T_{ij}:=\operatorname{Tr}[(\sigma_j\otimes C_i)\tau]$, a real 3×3 matrix.

Proposition 3 (Salvia, De Palma, and Giovannetti [16])**.** *The largest energy decrease of τ under unitaries on the qubit b is*

$$E^{\text{unit}}(\tau)=\operatorname{Tr}T+s_1+s_2-\operatorname{sgn}(\det T)\,s_3, \qquad (19)$$

where $s_1\geq s_2\geq s_3$ are the singular values of T.

This is the closed formula for the local ergotropy of a two-level system in Sec. III A of Ref. [16], their Eq. (19). Their matrix M (their Eq. (18)) is $-T^\top$ in the present notation. The argument is short. $U^\dagger\sigma_iU=\sum_j R_{ij}\sigma_j$ with $R\in SO(3)$, and every $R\in SO(3)$ arises. So the energy after the unitary is $\operatorname{Tr}[C_0\operatorname{Tr}_b\tau]+\sum_{ij}R_{ij}T_{ij}$. Minimizing over $SO(3)$ is the orthogonal Procrustes problem with a determinant constraint [22].

We apply Eq. (19) branch by branch, $E_B^{\text{unit}}=\sum_\mu p_\mu E^{\text{unit}}(\tau_\mu)$. We used 96 chain configurations: $L\in\{6,8,10,12\}$, $g\in\{0.5,1,2\}$, Alice at site 0, and Bob at every distance. On all of them E_B^{unit} equals the yield of the chain protocol's optimized conditional rotation [4] to 4.4×10^{-16} in absolute energy. The one-parameter rotation family of the protocol is therefore optimal among all unitaries on b. On the minimal model Eq. (19) reproduces $E_B^{\max}$ to 10^{-12}.

Unitaries are not Bob's best free operations in the ordered phase. At $g=0.5$ and $d=1$, a channel on b beats every unitary. At $L=6$ the Choi-matrix semidefinite program of Ref. [2] certifies 9.697×10^{-3} against 1.654×10^{-3}, a factor 5.86. A numerically optimized channel, a lower bound on the optimum, gives factors of 3.2, 5.86, 8.5, 10.5, 11.7, and 12.4 for $L=5,\ldots,10$. At $g=1$ and $g=2$ the rotation attains the single-site channel optimum to 2×10^{-7}, relative, on configurations with $E_B>10^{-6}$. That local channels can beat local unitaries is known [2, 16]; the Ising values are measurements, nothing more.

B. Entropic bounds, and why they are loose

Proposition 4 (entropic bounds)**.** *Let $\varepsilon_1\leq\varepsilon_2\leq\ldots$ be the eigenvalues of H_R and $H_{\min}(\tau):=-\ln\lambda_{\max}(\tau)$.*

(a) For unitaries on R or on any part of it,

$$E_B\leq\sum_\mu p_\mu\Big(\operatorname{Tr}[H_R\tau_\mu]-\kappa\big(H_R,H_{\min}(\tau_\mu)\big)\Big), \qquad (20)$$

where $\kappa(H,s):=\min\{\operatorname{Tr}[H\omega]:\omega\geq0,\operatorname{Tr}\omega=1,\omega\leq e^{-s}I\}$. This minimum equals $e^{-s}\sum_{i\leq n}\varepsilon_i+(1-ne^{-s})\varepsilon_{n+1}$, with $n=\lfloor e^s\rfloor$.

(b) Suppose the average $\rho_R=\sum_\mu p_\mu\tau_\mu$ is passive for Bob's unitaries on b, and Alice's outcome is binary with $p_\pm=\frac{1}{2}$. Then

$$E_B\leq 2c\Big(e^{-H_{\min}(X|R)}-\tfrac{1}{2}\Big), \qquad (21)$$

where $e^{-H_{\min}(X|R)}$ is the probability of guessing Alice's outcome X from R [23], and $c:=\frac{1}{2}\max_U\operatorname{spread}\big(H_R-(U^\dagger\otimes I)H_R(U\otimes I)\big)$ over unitaries U on b.

Bound (a) holds because $U\tau U^\dagger$ has the spectrum of τ and so is feasible for κ. Bound (b) combines passivity with the Hölder inequality and the Helstrom formula (Appendix A). The passivity hypothesis holds when the global state is strongly locally passive and Alice's instrument is energy-non-signalling; it was checked on every configuration.

Bound (a) is attained when the conditional states are pure and Bob acts on all of R. Then $H_{\min}=0$ and $\kappa=\varepsilon_1$, so it is the pure-state ergotropy. This happens in the minimal model, where the ratio is 1 to 1.5×10^{-11}.

Saturation there reflects purity, not a tight entropic principle. On the chains, where τ_μ is mixed, the bound is loose: E_B is at most 0.041 of it. Replacing $H_{\min}$ by its smoothed version [24] did not tighten it on any of 126 chain configurations. Bound (b) says that Bob extracts only what his region knows about Alice's outcome, and it vanishes exactly when the outcome is unguessable. But it is loose too: on the 96 configurations, E_B is at most 0.0080 of it.

The looseness has an identifiable cause. Alice's outcome moves τ_μ by an $O(1)$ trace distance, but in a nearly energy-neutral direction. The tight value (19) comes from a cancellation, inside the nuclear norm of T, between Bob's field row and the correlation rows, and norm-based bounds discard it. An entropic quantity that retains the cancellation is not known to us. Itoh, Masaki, and Matsueda derived information-thermodynamic bounds of a related kind for measurement-and-feedback extraction from pure states [25].

C. A daemonic gain

Francica *et al.* [26] define the daemonic ergotropy. It is the energy extractable from a system by unitaries conditioned on the outcome of a measurement on a correlated ancilla (their Eq. (3)); the daemonic gain is its excess over the unconditioned ergotropy (their Eq. (5)). Bernards *et al.* extended the notion to generalized measurements [27]. Read R as the system and the rest of the chain as the ancilla, and restrict the daemon's measurements to free instruments. Bob's optimal unitary yield is then a daemonic ergotropy. Without the message, Bob's unitary yield vanishes by strong local passivity, so E_B is entirely daemonic gain. Itoh *et al.* make the same connection between feedback extraction and daemonic ergotropy [25]. The resource theory adds a constraint on the daemon, not a new bound.

VIII. NUMERICAL CHECKS

A. The minimal model

On the grid of Fig. 1, Eq. (10) is computed from the Hamiltonian and its ground state. It agrees with Hotta's $E_B^{\max}$ to 3.7×10^{-15}, bound (20) is attained to 1.5×10^{-11} relative, and the local extractable energy is at most 2.3×10^{-15}. The last number is strong local passivity, certified by the semidefinite program of Ref. [2].

B. Random free operations

We applied 12 500 random free operations to random pure, mixed, ground-state-mixture, and thermal states. Of these, 10 550 acted on systems of 2 to 6 qubits, in seven cases: the minimal model at three values of (h, k), and Ising chains with $L = 4$–6 and Alice regions of one or two sites. The operations were commuting-Kraus instruments with 1 to 3 Kraus operators, channels on B, and unitaries on B. The other 1 950 acted on harmonic chains of 2 to 4 modes (Sec. VIII E). There the operations were Gaussian unitaries on A that commute with the bond quadratures, symplectic maps on B, and loss on B. The worst violation of Eqs. (13) and (14), and of negativity monotonicity, was 2.1×10^{-14}. The energy-signalling defect of the sampled instruments was at most 1.8×10^{-14}.

The checks can fail. In every case a random non-free instrument on A raised $W_\to$, by 0.96 to 7.72 on qubits and by 0.52 to 2.48 on modes. An energy-injecting channel on B raised it by 2.19 to 7.50, but never by more than the injected energy, to within 9.2×10^{-15}.

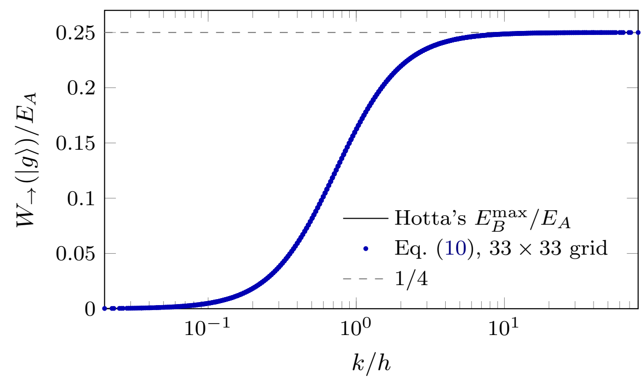

FIG. 1. The minimal model. The potential (10) of the ground state, computed from the Hamiltonian on a logarithmic 33×33 grid ($h \in [0.05, 5]$, $k \in [0.1, 4]$) and divided by Alice's injected energy E_A, collapses onto a function of k/h. It agrees with Hotta's closed form to 3.7×10^{-15} and stays below 1/4 [Eq. (12)]; the grid maximum is 0.249978 at $(h, k) = (0.05, 4)$.

C. Two-way protocols

On the minimal model we computed the optimal finite-round two-way value with Bob-only cash-out. The computation recurses over which party measures in each round, for up to three rounds, and the result equals $W_\to$ to 2×10^{-15}. On chains the recursion would need Bob's full channel optimum on entangled states, so there the evidence is sampled instead. We ran 1 180 random two-way protocols on the minimal model and on Ising chains of 4 to 6 sites. Each is a sequence of three moves. A move is a commuting-Kraus instrument on A conditioned on the record so far, an instrument on B whose outcome Alice may use, or a cash-out on B. Across them the ledger of Corollary 1 is conserved to 9.0×10^{-15}, while a non-free instrument on A breaks it by at least 1.08.

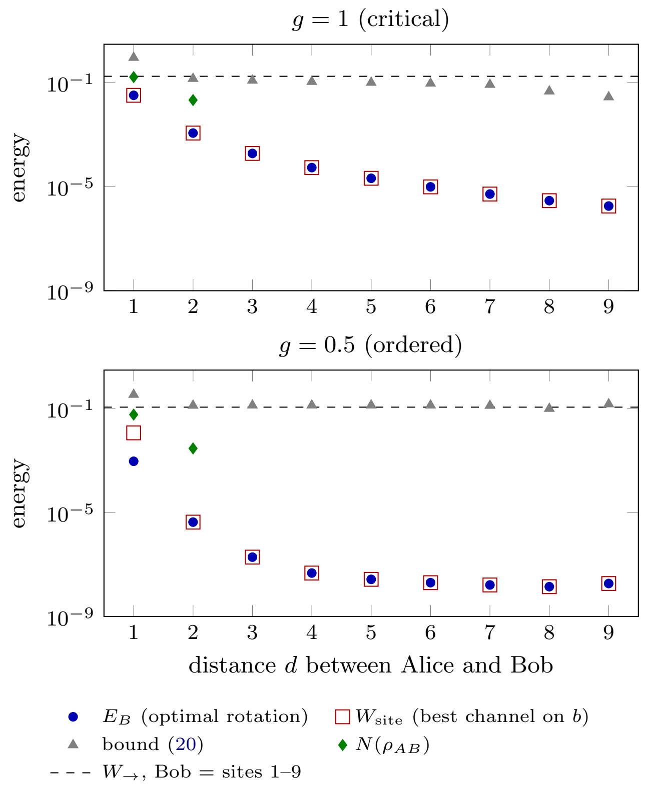

FIG. 2. An Ising chain of $L = 10$ sites; Alice at site 0 measures σ_x, and Bob holds the site at distance d. E_B is the yield of the optimized conditional rotation, which equals the exact unitary yield (19). W_{site} is the best single-site channel, found numerically. The triangles show bound (20) on the region R. $N(\rho_{AB})$ is the two-site negativity of the ground state; zeros are not drawn. The dashed line is the potential with Bob holding the whole complement: 0.1759 at $g = 1$ and 0.1124 at $g = 0.5$, against $E_A = 0.851$ and 0.242. In the ordered phase the best channel beats the rotation at $d = 1$ by a factor of 12.4.

D. The chains

Figure 2 shows an Ising chain of 10 sites; the data cover $L = 4$–12 and $g \in \{0.5, 1, 2\}$. Bob's single-site yield is at most 0.686 of the potential with Bob holding the whole complement; the maximum occurs at $g = 2$, $d = 1$. At $L = 12$ the potential (10) and the region quantities are computed without forming dense $2^L \times 2^L$ matrices, and they agree with the dense computation to 10^{-9} where both run.

E. A Gaussian analogue

For a harmonic chain $H = \frac{1}{2}(p \cdot p + x \cdot Kx)$ with Bob holding the complement, the bond terms are $x_a K_{ab} x_b$. Homodyne detection of Alice's bond quadratures x_a commutes with them. Given the outcome vector ξ, Bob's conditional state is Gaussian, and his effective Hamiltonian acquires the linear term $\xi^\top K_{AB} x_B$. That lowers its ground energy by $\frac{1}{2}\xi^\top K_{AB} K_{BB}^{-1} K_{AB}^\top \xi$. Averaging over ξ gives the analogue of Eq. (10),

$$W = \langle H_B^{\text{loc}} \rangle - E_0(K_{BB}) + \tfrac{1}{2} \operatorname{Tr}\left[K_{AB} K_{BB}^{-1} K_{AB}^\top \Sigma\right], \quad (22)$$

where Σ is the covariance of the bond quadratures and $E_0(K_{BB})$ is the ground energy of Bob's uncoupled chain. For 2 to 4 modes it agrees with the explicit conditional-state integral to 6.0×10^{-16}, with W ranging from 0.0017 to 0.045 over the masses studied. Equation (22) is the value of the homodyne protocol; we prove no Gaussian or infinite-dimensional analogue of Theorems 2 and 3.

IX. RELATED WORK

Local extraction. Strong local passivity was characterized in Refs. [1, 2]. Alhambra *et al.* cast local channel extraction as a semidefinite program. Salvia, De Palma, and Giovannetti introduced the local ergotropy [16], whose qubit formula is our Proposition 3. Castellano *et al.* extended it by exploiting the free evolution of system and environment [28]. Bhattacharyya *et al.* studied maps that are not completely positive [29]. Ergotropy itself is due to Allahverdyan, Balian, and Nieuwenhuizen [30].

Quantum energy teleportation. The protocol and its spin-chain and field-theory versions are Hotta's [3–5], including energy–entanglement relations for the minimal model [21]. Ref. [8] reviews experiments and applications. Wang and Yao proved a trade-off between energy and information teleportation [14]. Haque derived an upper bound on the energy extractable in gapped lattice systems with a unique ground state that decays exponentially with the distance [31]. Itoh *et al.* derived information-thermodynamic bounds [25]. Yan, Zhang, and Wang studied QET on mixed steady states prepared by environments [10]. Xie, Sajjan, and Kais proposed storing the extracted energy in a quantum register [32], which is Bob's device made quantum.

Closest work. Xie, Sajjan, and Kais [11] relax three conventional constraints on QET: a ground state, a measurement that commutes with the interaction, and entanglement. They introduce Bob's local effective Hamiltonian, which is our $H^{(a)}$. In their protocol Alice measures σ_x^A while the coupling contains $\sigma_y^A \sigma_y^B$. That measurement violates Eq. (3), since $\bar{\mathcal{A}}^\dagger(\sigma_y^A) = 0$, so it is not free here, although on their initial state it injects no energy into the coupling. Their bond operators σ_x^A and σ_y^A do not commute, so the model lies outside the scope of Proposition 1 and Theorem 2. Condition (3) is state-independent. For commuting bonds, Theorem 2 shows it is exactly the commutation constraint whenever the bond spectrum is in convex position. Wu *et al.* [9] find strongly locally passive pure states of the minimal model by a correlation-matrix analysis, with Bob's yield exceeding Alice's injection for some non-ground states. Fan *et al.* [12] analyse QET with the correlation matrix, identify quantum steering as its essence, and define "QET passive" states. Those states are the analogue of our free states; their class of allowed operations may differ. Fan *et al.* [13] study how quantum resources affect the efficiency of QET and give necessary and sufficient conditions for the minimal model. These correlation-matrix analyses may contain the two-qubit cases of Propositions 1 and 3. We claim neither: Proposition 3 is Salvia *et al.*'s, and in the minimal model Proposition 1 gives Hotta's value. As far as we know, none of these works formulates a class of free operations with a ledger monotone, characterizes the free class, or addresses two-way communication.

X. DISCUSSION AND LIMITATIONS

The theory's central object is a linear functional. For commuting bonds in convex position, $W_\to(\rho) = \mathrm{Tr}[W_{\mathrm{op}}\rho]$, and everything Bob can receive by free means, with one-way or two-way communication, is the expectation of one operator. The limitations are the following.

- Proposition 1 and Theorems 2 and 3 assume commuting bond operators. Non-commuting bonds admit no finest free measurement, and we do not treat them.
- Off convex position we exhibit one free instrument outside the commuting class, not the optimal one. Whether two-way communication helps there is open.
- The theory credits Bob's side only. Alice's apparatus pays E_A, and crediting a second device buys nothing net (Table II).
- For a single-qubit Bob, Eq. (19) is exact for unitaries, while the channel optimum is a semidefinite program [2]. For a multi-qubit Bob we have no closed form.
- The entropic bounds are valid but loose by one to two orders of magnitude on the chains.
- The numerics are finite-size: at most 12 sites and 4 modes. Nothing here is a statement about a thermodynamic limit or a quantum field.

ACKNOWLEDGMENTS

AI assistants, including Anthropic's Claude models, were used extensively in this work: to write and test the analysis code, to run the numerical checks, to search the literature, and to draft this manuscript. The author directed the work and takes full responsibility for its content.

Appendix A: Proofs

Theorem 1(a). Fix $\epsilon > 0$. For each μ choose a free instrument $\{K'_{\nu|\mu}\}$ and channels $\{\mathcal{E}_{\mu\nu}\}$ that deliver at least $W_\to(\rho_\mu) - \epsilon$ from ρ_μ. The composed instrument $\{K'_{\nu|\mu}K_\mu\}$ is free, and with the channels $\mathcal{E}_{\mu\nu}$ it delivers $\sum_\mu p_\mu[W_\to(\rho_\mu) - \epsilon]$ from ρ. Hence $W_\to(\rho) \geq \sum_\mu p_\mu W_\to(\rho_\mu) - \epsilon$ for every ϵ.

Theorem 1(b). Let $\rho' = (\mathrm{id} \otimes \mathcal{E})(\rho)$, fix $\epsilon > 0$, and choose $(\{K_\mu\}, \{\mathcal{E}_\mu\})$ delivering at least $W_\to(\rho') - \epsilon$ from ρ'. Run the same instrument on ρ, with Bob's channels $\mathcal{E}_\mu \circ \mathcal{E}$. Because K_μ acts on A and $\mathcal{E}$ on B, $(K_\mu \otimes I)\rho'(K_\mu \otimes I)^\dagger = (\mathrm{id} \otimes \mathcal{E})(\tilde{\rho}_\mu)$, so the final energies of the two runs coincide. By Eq. (3) their initial energies are $\mathrm{Tr}[H_B^{\mathrm{loc}}\rho]$ and $\mathrm{Tr}[H_B^{\mathrm{loc}}\rho']$, which differ by $w_B(\mathcal{E}, \rho)$. Hence $W_\to(\rho) \geq W_\to(\rho') - \epsilon + w_B(\mathcal{E}, \rho)$. The same arguments apply verbatim to $W_\to^{\mathrm{CK}}$.

Corollary 1. Bob's steps obey Eq. (14) and raise D by w_B. Alice's steps obey Eq. (13) and leave D unchanged. Communication changes nothing. Mixing and discarding flags cannot increase $W_\to$, by convexity. Since $W_\to \geq 0$, the final D is at most $W_\to(\rho)$.

Theorem 2. ($\Rightarrow$) Let every $x(b)$ be extreme, and let v be a unit vector in the range of Π_b. By Eq. (15), $p_a := \langle v|M_{ab}|v\rangle$ is a probability distribution with $\sum_a p_a x(a) = x(b)$. Suppose $p_b < 1$. Then $x(b) = \sum_{a\neq b} q_a x(a)$ with $q_a = p_a/(1 - p_b)$, and taking any a_0 with $q_{a_0} > 0$ writes $x(b)$ as a combination of $x(a_0) \neq x(b)$ and a point of the hull. That is either trivial, which would give $x(a_0) = x(b)$, or contradicts extremality. So $p_b = 1$ and $p_a = 0$ for $a \neq b$. Since $M_{ab} \geq 0$ is supported in the range of Π_b and v was arbitrary, $M_{ab} = 0$, i.e., $\sum_\mu \|\Pi_a K_\mu \Pi_b \psi\|^2 = 0$ for all ψ. Every K_μ is then block diagonal and commutes with every Π_a, hence with every X_j. ($\Leftarrow$) If some $x(b)$ is not extreme, it lies in $\mathrm{conv}\{x(a)\}_{a\neq b}$, because the extreme points of the hull of a finite set belong to the set. The instrument (16) is then complete, satisfies Eq. (3), and has $\Pi_a K_a \Pi_b \neq 0$.

Proposition 2. Within the proposition's restriction the potential is $W_\to^{\mathrm{CK}}$, so $Q = W_\to^{\mathrm{CK}} - W_{\mathrm{loc}}$ (in the minimal model $W_\to^{\mathrm{CK}} = W_\to$). Write $m(\rho) := \min_{\mathcal{E}} \mathrm{Tr}[H_B^{\mathrm{loc}}(\mathrm{id} \otimes \mathcal{E})(\rho)]$, so that $W_{\mathrm{loc}}(\rho) = \mathrm{Tr}[H_B^{\mathrm{loc}}\rho] - m(\rho)$, and $\Lambda(\rho) := \sum_a p_a \lambda_{\min}(H^{(a)})$, so that $W_\to^{\mathrm{CK}}(\rho) = \mathrm{Tr}[H_B^{\mathrm{loc}}\rho] - \Lambda(\rho)$ by Eq. (10). Hence $Q = m(\rho) - \Lambda(\rho)$. (a) A channel $\mathcal{E}'$ on B leaves the p_a and hence Λ unchanged, and $m(\mathcal{E}'(\rho)) \geq m(\rho)$ because $\mathcal{E} \circ \mathcal{E}'$ is a channel. So Q does not decrease. For a commuting-Kraus instrument, $\sum_\mu K_\mu^\dagger \Pi_a K_\mu = \Pi_a$ gives $\sum_\mu p_\mu \Lambda(\rho_\mu) = \Lambda(\rho)$. Let $\mathcal{E}^\star$ attain $m(\rho)$. Then $\sum_\mu p_\mu m(\rho_\mu) \leq \sum_\mu \mathrm{Tr}[H_B^{\mathrm{loc}}(\mathrm{id} \otimes \mathcal{E}^\star)(\tilde{\rho}_\mu)] = m(\rho)$ by Eq. (3). So Q does not increase on average. (b) For a product state, $m = \lambda_{\min}(\bar{H})$ with the mean field $\bar{H} = h\sigma_z + 2k\langle\sigma_x^A\rangle\sigma_x$. For ρ_1: $W_\to = W_{\mathrm{loc}} = h + \sqrt{h^2 + 4k^2}$. For ρ_2: $W_\to = h + \sqrt{h^2 + 4k^2}$, since Alice's σ_x measurement restores the field $\pm 2k\sigma_x$ on each branch, while $W_{\mathrm{loc}} = 2h$. (c) For ρ_3, Bob can measure σ_z and prepare the ground state of $H^{(\pm)}$, reaching $m = \Lambda = -\sqrt{h^2 + 4k^2}$, so $Q = 0$. $\mathrm{Tr}[H_B^{\mathrm{loc}}\rho_3] = 0$ and $\mathrm{Tr}[H_B^{\mathrm{loc}}\rho_5] = -h$, so the reset delivers h. For the product state ρ_5, $W_\to = \sqrt{h^2 + 4k^2} - h$ and $W_{\mathrm{loc}} = 0$.

Proposition 4. (a) $U\tau U^\dagger$ has largest eigenvalue $e^{-H_{\min}(\tau)}$, so it is feasible in the definition of κ. The minimum is attained on states diagonal in the eigenbasis of H by filling the lowest levels to the cap e^{-s}. (b) Let $Y_U := H_R - (U^\dagger \otimes I)H_R(U \otimes I)$. Passivity gives $\mathrm{Tr}[Y_U\rho_R] \le 0$ for every U. So $E_B = \sum_\mu p_\mu \max_U \mathrm{Tr}[Y_U\tau_\mu] \le \sum_\mu p_\mu \max_U \mathrm{Tr}[Y_U(\tau_\mu - \rho_R)]$. For traceless δ, $|\mathrm{Tr}[Y\delta]| \le \frac{1}{2}\,\mathrm{spread}(Y)\|\delta\|_1$, so this is at most $c\sum_\mu p_\mu\|\tau_\mu - \rho_R\|_1 = \frac{c}{2}\|\tau_+ - \tau_-\|_1$. The Helstrom formula $e^{-H_{\min}(X|R)} = \frac{1}{2} + \frac{1}{4}\|\tau_+ - \tau_-\|_1$ [23] gives Eq. (21).

Appendix B: Code and data

Every number in this paper comes from archived data produced by one script. The script is `papers/resource-theory-vacuum-manipulation/notebook.py` (SHA-256 `ddd18909...e68f`), run on 5 September 2026 on a tree whose full test suite passed. The candidate's own tests are in `tests/test_resource_theory.py` (44 tests). The figure tables are written from the archive by `export_figdata.py` next to this manuscript. Three statements were checked separately while this draft was written: the certified gap at $L = 6$ (Sec. VII), the identity of Eq. (11) with Hotta's formula, and Proposition 2(c). The code and data are available at `https://github.com/mruckman1/vacuum_energy`. Its numbering differs from the paper's, as follows.

TABLE III. Correspondence between this paper and the module docstring of `vacuum/qet/resource.py`.

This paper	Code
Eq. (3), energy non-signalling	Eq. (2)
Eq. (8), the potential	Eq. (3)
Proposition 1	Proposition A, Eq. (4)
Theorem 1	Theorem 1, Eqs. (5a)–(5b)
Theorem 2, Eqs. (15)–(16)	Theorem 2, Eqs. (7)–(8)
Eq. (17)	Eq. (20)
Theorem 3	Theorem 4, Eqs. (10)–(11)
Corollary 2	Eq. (12)
Proposition 2	surplus section
Proposition 3	Proposition B, Eq. (15)
Proposition 4(a)	chain (ii)
Proposition 4(b)	Proposition C, Eq. (17)
Eq. (22)	Gaussian mirror

[1] M. Frey, K. Funo, and M. Hotta, Strong local passivity in finite quantum systems, Phys. Rev. E **90**, 012127 (2014).

[2] Á. M. Alhambra, G. Styliaris, N. A. Rodríguez-Briones, J. Sikora, and E. Martín-Martínez, Fundamental limitations to local energy extraction in quantum systems, Phys. Rev. Lett. **123**, 190601 (2019).

[3] M. Hotta, Quantum measurement information as a key to energy extraction from local vacuums, Phys. Rev. D **78**, 045006 (2008).

[4] M. Hotta, Quantum energy teleportation in spin chain systems, J. Phys. Soc. Jpn. **78**, 034001 (2009).

[5] M. Hotta, Quantum energy teleportation: An introductory review (2011), arXiv:1101.3954 [quant-ph].

[6] N. A. Rodríguez-Briones, H. Katiyar, E. Martín-Martínez, and R. Laflamme, Experimental activation of strong local passive states with quantum information, Phys. Rev. Lett. **130**, 110801 (2023).

[7] K. Ikeda, Demonstration of quantum energy teleportation on superconducting quantum hardware, Phys. Rev. Appl. **20**, 024051 (2023).

[8] B. Ragula and E. Martín-Martínez, A review of applications of quantum energy teleportation: from experimental tests to thermodynamics and spacetime engineering, Can. J. Phys. **103**, 1094 (2025).

[9] F.-L. Wu, H. Fan, L. Wang, S.-Q. Liu, S.-Y. Liu, and W.-L. Yang, Strong local passive state in the minimal quantum energy teleportation model, Phys. Rev. A **109**, 062208 (2024).

[10] X. Yan, K. Zhang, and J. Wang, Quantum energy teleportation on mixed steady states prepared by equilibrium and nonequilibrium environments, Phys. Lett. A **591**, 131881 (2026).

[11] S. Xie, M. Sajjan, and S. Kais, Strong local passivity in unconventional scenarios: A new protocol for amplified quantum energy teleportation, Entropy **27**, 1147 (2025).

[12] H. Fan, F.-L. Wu, L. Wang, S.-Q. Liu, and S.-Y. Liu, Strong quantum energy teleportation, Phys. Rev. A **110**, 052424 (2024).

[13] H. Fan, F.-L. Wu, L. Wang, S.-Q. Liu, and S.-Y. Liu, The role of quantum resources in quantum energy teleportation, Quantum Inf. Process. **23**, 367 (2024).

[14] J. Wang and S. Yao, Quantum energy teleportation versus information teleportation, Quantum **8**, 1564 (2024).

[15] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).

[16] R. Salvia, G. De Palma, and V. Giovannetti, Optimal local work extraction from bipartite quantum systems in the presence of Hamiltonian couplings, Phys. Rev. A **107**, 012405 (2023).

[17] P. Pfeuty, The one-dimensional Ising model with a transverse field, Ann. Phys. **57**, 79 (1970).

[18] A. Arias, A. Gheondea, and S. Gudder, Fixed points of quantum operations, J. Math. Phys. **43**, 5872 (2002).

[19] W. Pusz and S. L. Woronowicz, Passive states and KMS states for general quantum systems, Commun. Math. Phys. **58**, 273 (1978).

[20] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[21] M. Hotta, Energy-entanglement relation for quantum energy teleportation, Phys. Lett. A **374**, 3416 (2010).

[22] W. Kabsch, A solution for the best rotation to relate two sets of vectors, Acta Crystallogr. A **32**, 922 (1976).

[23] R. König, R. Renner, and C. Schaffner, The operational meaning of min- and max-entropy, IEEE Trans. Inf. Theory **55**, 4337 (2009).

[24] M. Tomamichel, *Quantum Information Processing with Finite Resources*, SpringerBriefs in Mathematical Physics (Springer, Cham, 2016).

[25] K. Itoh, Y. Masaki, and H. Matsueda, Upper bound on locally extractable energy from entangled pure state under feedback control, Phys. Rev. E **113**, 024108 (2026).

[26] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: enhanced work extraction from quantum correlations, npj Quantum Inf. **3**, 12 (2017).

[27] F. Bernards, M. Kleinmann, O. Gühne, and M. Paternostro, Daemonic ergotropy: Generalised measurements and multipartite settings, Entropy **21**, 771 (2019).

[28] R. Castellano, D. Farina, V. Giovannetti, and A. Acín, Extended local ergotropy, Phys. Rev. Lett. **133**, 150402 (2024).

[29] A. Bhattacharyya, K. Sen, and U. Sen, Noncompletely positive quantum maps enable efficient local energy extraction in batteries, Phys. Rev. Lett. **132**, 240401 (2024).

[30] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, Europhys. Lett. **67**, 565 (2004).

[31] T. Haque, Nontrivial bounds on extractable energy in quantum energy teleportation for gapped manybody systems with a unique ground state, Phys. Lett. A **584**, 131613 (2026).

[32] S. Xie, M. Sajjan, and S. Kais, Extracting and storing energy from a quasi-vacuum on a quantum computer (2024), arXiv:2409.03973 [quant-ph].